

HIGHER ORDER RIESZ TRANSFORMS IN THE ULTRASPHERICAL SETTING AS PRINCIPAL VALUE INTEGRAL OPERATORS

JORGE J. BETANCOR, JUAN C. FARÍÑA, LOURDES RODRÍGUEZ-MESA, AND RICARDO TESTONI

ABSTRACT. In this paper we represent the k -th Riesz transform in the ultraspherical setting as a principal value integral operator for every $k \in \mathbb{N}$. We also measure the speed of convergence of the limit by proving L^p -boundedness properties for the oscillation and variation operators associated with the corresponding truncated operators.

1. INTRODUCTION

Muckenhoupt and Stein [10] introduced a notion of conjugate functions associated with ultraspherical expansions. In this setting the conjugate function appears as a boundary value of a conjugate harmonic extension associated with a suitable Cauchy-Riemann type equations.

Assume that $\lambda > 0$. For every $n \in \mathbb{N}$, we denote by P_n^λ the ultraspherical polynomial of degree n ([14]). These polynomials are defined by the generating relation

$$(1 - 2tw + w^2)^{-\lambda} = \sum_{k=0}^{\infty} w^k P_k^\lambda(t).$$

The sequence $\{P_n^\lambda(\cos \theta)\}_{n \in \mathbb{N}}$ is orthogonal and complete in the space $L^2((0, \pi), dm_\lambda(\theta))$, where $dm_\lambda(\theta) = (\sin \theta)^{2\lambda} d\theta$. When $2\lambda = k - 2$, with $k \in \mathbb{N}$, the λ -ultraspherical polynomial P_n^λ , $n \in \mathbb{N}$, arises in the Fourier analysis of functions in the surface of the n -Euclidean space sphere that are invariant under the rotations leaving a given axis fixed.

For every $n \in \mathbb{N}$, $P_n^\lambda(\cos \theta)$ is an eigenfunction of the operator

$$L_\lambda = -\frac{d^2}{d\theta^2} - 2\lambda \cot \theta \frac{d}{d\theta} + \lambda^2,$$

associated with the eigenvalue $\mu_n = (n + \lambda)^2$. The operator L_λ can be written as follows

$$L_\lambda = -\left(\frac{d}{d\theta}\right)^* \frac{d}{d\theta} + \lambda^2,$$

Date: May 11, 2010.

2000 *Mathematics Subject Classification.* 42C05 (primary), 42C15 (secondary).

This paper is partially supported by MTM2007/65609.

where $\left(\frac{d}{d\theta}\right)^* = \frac{d}{d\theta} + 2\lambda \cot \theta$ denotes the formal adjoint of $\frac{d}{d\theta}$ in $L^2((0, \pi), dm_\lambda(\theta))$.

In [1] Buraczewski, Martínez, Torrea and Urban defined a Riesz transform in the ultraspherical setting associated to L_λ . Note that this operator L_λ is slightly different than the one considered by Muckenhoupt and Stein (see [10, p. 23]). In [1] the authors follow the ideas developed in the monography of Stein [13].

Suppose that $f \in L^2((0, \pi), dm_\lambda(\theta))$. The ultraspherical expansion of f is

$$f(\theta) = \sum_{n=0}^{\infty} a_n^\lambda(f) \frac{P_n^\lambda(\cos \theta)}{\|P_n^\lambda(\cos \cdot)\|_{L^2((0, \pi), dm_\lambda(\theta))}},$$

where, for every $n \in \mathbb{N}$,

$$a_n^\lambda(f) = \int_0^\pi f(\theta) \frac{P_n^\lambda(\cos \theta)}{\|P_n^\lambda(\cos \cdot)\|_{L^2((0, \pi), dm_\lambda(\theta))}} dm_\lambda(\theta).$$

The Poisson integral $P_t^\lambda(f)$, $t > 0$, is given by

$$P_t^\lambda(f)(\theta) = e^{-t\sqrt{L_\lambda}} f(\theta) = \sum_{n=0}^{\infty} a_n^\lambda(f) e^{-t(n+\lambda)} \frac{P_n^\lambda(\cos \theta)}{\|P_n^\lambda(\cos \cdot)\|_{L^2((0, \pi), dm_\lambda(\theta))}}, \quad t > 0.$$

According to [10, (2.12)] we can write

$$(1) \quad P_t^\lambda f(\theta) = \int_0^\pi r^\lambda P_\lambda(e^{-t}, \theta, \varphi) f(\varphi) dm_\lambda(\varphi), \quad t > 0,$$

where, for each $0 < r < 1$ and $\theta, \varphi \in (0, \pi)$,

$$P_\lambda(r, \theta, \varphi) = \frac{\lambda}{\pi} (1 - r^2) \int_0^\pi \frac{\sin^{2\lambda-1} t}{(1 - 2r(\cos \theta \cos \varphi + \sin \theta \sin \varphi \cos t) + r^2)^{\lambda+1}} dt.$$

The L^p -boundedness properties for these Poisson integrals and the corresponding maximal operator were established in [1, Theorem 2.4] (see also [10, Theorem 2]). For every $\alpha > 0$, the fractional power $L_\lambda^{-\alpha}$ of the operator L_λ is defined by

$$L_\lambda^{-\alpha} f(\theta) = \frac{1}{\Gamma(2\alpha)} \int_0^\infty e^{-t\sqrt{L_\lambda}} f(\theta) t^{2\alpha-1} dt, \quad f \in L^2((0, \pi), dm_\lambda(\theta)).$$

By using (1) we get, for every $\alpha > 0$ and $f \in L^2((0, \pi), dm_\lambda(\theta))$,

$$\begin{aligned} L_\lambda^{-\alpha} f(\theta) &= \frac{1}{\Gamma(2\alpha)} \int_0^\infty \int_0^\pi P^\lambda(e^{-t}, \theta, \varphi) f(\varphi) dm_\lambda(\varphi) t^{2\alpha-1} dt \\ &= \int_0^\pi f(\varphi) \frac{1}{\Gamma(2\alpha)} \int_0^1 P^\lambda(r, \theta, \varphi) \left(\log \frac{1}{r}\right)^{2\alpha-1} \frac{1}{r} dr dm_\lambda(\varphi). \end{aligned}$$

Following [13] the Riesz transform of order $k \in \mathbb{N}$, R_λ^k , is defined as

$$R_\lambda^k f = \frac{d^k}{d\theta^k} L_\lambda^{-\frac{k}{2}} f,$$

when f is a nice function (for instance, $f \in \text{span}\{P_n^\lambda(\cos \theta)\}_{n \in \mathbb{N}}$ or f is a smooth function with compact support on $(0, \pi)$).

It was proved in [1, Theorem 2.14] (when $k = 1$) and [2, Theorem 1.4] (when $k > 1$) that the operator R_λ^k can be extended to $L^p((0, \pi), w(\theta)dm_\lambda(\theta))$ as a bounded operator from $L^p((0, \pi), w(\theta)dm_\lambda(\theta))$ into itself, when $1 < p < \infty$ and $w \in A_\lambda^p$, and as a bounded operator from $L^1((0, \pi), w(\theta)dm_\lambda(\theta))$ into $L^{1,\infty}((0, \pi), w(\theta)dm_\lambda(\theta))$, when $w \in A_\lambda^1$. Here, for every $1 \leq p < \infty$, by A_λ^p we denote the Muckenhoupt class of weights associated with the doubling measure $dm_\lambda(\theta)$ on $(0, \pi)$.

In this paper we prove that the k -th Riesz transform R_λ^k is a principal value integral operator, for every $k \in \mathbb{N}$. We extend [1, Theorem 2.13] where the result is shown for $k = 1$.

Theorem 1.1. *Let $\lambda > 0$ and $k \in \mathbb{N}$. For every $1 \leq p < \infty$ and $w \in A_\lambda^p$, we have that if $f \in L^p((0, \pi), w(\theta)dm_\lambda(\theta))$*

$$(2) \quad R_\lambda^k f(\theta) = \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta - \varphi| > \varepsilon}^\pi R_\lambda^k(\theta, \varphi) f(\varphi) dm_\lambda(\varphi) + \gamma_k f(\theta), \quad \text{a.e. } \theta \in (0, \pi),$$

where

$$R_\lambda^k(\theta, \varphi) = \frac{1}{\Gamma(k)} \int_0^1 \frac{\partial^k}{\partial \theta^k} P_\lambda(r, \theta, \varphi) \left(\log \frac{1}{r} \right)^{k-1} r^{\lambda-1} dr, \quad \theta, \varphi \in (0, \pi),$$

and $\gamma_k = 0$, when k is odd, and $\gamma_k = (-1)^{\frac{k}{2}}$, when k is even.

The complete proof of this theorem is presented in Section 2. It is a crucial point in the proof the estimates established in Lemma 2.1 below. We prove in this lemma that in the local region, that is, close to the diagonal $\{\theta = \varphi\}$, the kernel $R_\lambda^k(\theta, \varphi)$ differs from the kernel of the k -th Euclidean Riesz transform by an integrable function. Also, we show that far from the diagonal $R_\lambda^k(\theta, \varphi)$ is bounded by Hardy type kernels.

Suppose that $\{T_\varepsilon\}_{\varepsilon > 0}$ is a family of operators defined on $L^p(\Omega, \mu)$, for some measure space (Ω, μ) and $1 \leq p < \infty$, such that for every $f \in L^p(\Omega, \mu)$ there exists $\lim_{\varepsilon \rightarrow 0^+} T_\varepsilon f(x)$, μ -a.e. $x \in \Omega$. It is an interesting question to measure the speed of that convergence. In order to do this it is usual to analyze expressions involving differences like $|T_\varepsilon f - T_\eta f|$, $\varepsilon, \eta > 0$. The oscillation and variation operators defined as follows have been used for this purpose. The oscillation operator associated with $\{T_\varepsilon\}_{\varepsilon > 0}$ is defined by

$$O(\{T_\varepsilon\})(f)(x) = \left(\sum_{i=0}^{\infty} \sup_{t_{i+1} \leq \varepsilon_{i+1} < \varepsilon_i < t_i} |T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)|^2 \right)^{\frac{1}{2}},$$

for a fixed real sequence $\{t_i\}_{i \in \mathbb{N}}$ decreasing to zero. For every $\rho > 2$ the ρ -variation operator for $\{T_\varepsilon\}_{\varepsilon > 0}$ is given as follows

$$V_\rho(\{T_\varepsilon\})(f)(x) = \sup_{\{\varepsilon_i\}_{i \in \mathbb{N}}} \left(\sum_{i=0}^{\infty} |T_{\varepsilon_{i+1}}f(x) - T_{\varepsilon_i}f(x)|^\rho \right)^{\frac{1}{\rho}},$$

where the supremum is taken over all real sequences $\{\varepsilon_i\}_{i \in \mathbb{N}}$ decreasing to zero. These operators appear in an ergodic context. In [3], [4] and [8] the L^p -boundedness properties for the oscillation and variation operators were studied when T_ε , $\varepsilon > 0$, represents the truncated Hilbert and higher dimensional Riesz transform, and Euclidean Poisson semigroup (see also [6] and the references therein). The corresponding results for the truncated ultraspherical Riesz transform were established in [1, Theorem 8.3]. We also measure the speed of convergence in (2) in terms of variation and oscillation operators for the corresponding truncated operators. Next result is an extension of [1, Theorem 8.3] for the higher Riesz transform R_λ^k .

Theorem 1.2. *Let $\lambda > 0$ and $k \in \mathbb{N}$. For every $\varepsilon > 0$ we define by $R_{\lambda,\varepsilon}^k$ the ε -truncation of R_λ^k as follows*

$$R_{\lambda,\varepsilon}^k(f)(\theta) = \int_{0, |\theta-\varphi| > \varepsilon}^{\pi} R_\lambda^k(\theta, \varphi) f(\varphi) dm_\lambda(\varphi).$$

If $\{t_i\}_{i \in \mathbb{N}}$ is a real decreasing sequence that converges to zero, the oscillation operator $O(\{R_{\lambda,\varepsilon}^k\})$ is a bounded operator from $L^p((0, \pi), dm_\lambda(\varphi))$ into itself, for every $1 < p < \infty$, and from $L^1((0, \pi), dm_\lambda(\varphi))$ to $L^{1,\infty}((0, \pi), dm_\lambda(\varphi))$. Also, for every $\rho > 2$, the variation operator $V_\rho(\{R_{\lambda,\varepsilon}^k\})$ is bounded from $L^p((0, \pi), dm_\lambda(\varphi))$ into itself, for every $1 < p < \infty$, and from $L^1((0, \pi), dm_\lambda(\varphi))$ to $L^{1,\infty}((0, \pi), dm_\lambda(\varphi))$.

We remark that the representation of the k -th Riesz transform R_λ^k as a principal value integral operator will allow us to investigate weighted norm inequalities for R_λ^k involving a class of weights wider than the Muckenhoupt class considered in [2]. This question will be studied in a forthcoming paper.

Troughout this paper by C we always denote a positive constant that can change from one line to the other one and i, j represent nonnegative integers.

2. PROOF OF THEOREM 1.1

In [2, Theorem 1.5] it was established that, for every $k \in \mathbb{N}$, the k -th Riesz transform R_λ^k is a Calderón-Zygmund operator in the homogeneous type space $((0, \pi), |\cdot|, dm_\lambda(\theta))$. Then,

according to [7, Theorem 9.4.5] the maximal operator $R_{\lambda,*}^k$ given by

$$R_{\lambda,*}^k(f) = \sup_{\varepsilon > 0} |R_{\lambda,\varepsilon}^k(f)|,$$

where $R_{\lambda,\varepsilon}^k$ is defined as in Theorem 1.2, is bounded from $L^p((0, \pi), w(\theta)dm_\lambda(\theta))$ into itself, when $1 < p < \infty$ and $w \in A_\lambda^p$, and from $L^1((0, \pi), w(\theta)dm_\lambda(\theta))$ into $L^{1,\infty}((0, \pi), w(\theta)dm_\lambda(\theta))$, when $w \in A_\lambda^1$. Suppose we have proved that, for every $f \in C_c^\infty(0, \pi)$, there exists the limit

$$(3) \quad T_\lambda^k(f)(\theta) = \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta-\varphi| > \varepsilon}^{\pi} R_\lambda^k(\theta, \varphi) f(\varphi) dm_\lambda(\varphi), \quad \text{a.e. } \theta \in (0, \pi),$$

and that $T_\lambda^k f = R_\lambda^k f - \gamma_k f$. Then, L^p -boundedness properties of the maximal operator $R_{\lambda,*}^k$ imply that the limit in (3) exists for almost all $\theta \in (0, \pi)$, for every $f \in L^p((0, \pi), w(\theta)dm_\lambda(\theta))$, $1 \leq p < \infty$, and $w \in A_\lambda^p$. Moreover, by defining T_λ^k in the obvious way on $L^p((0, \pi), w(\theta)dm_\lambda(\theta))$, $1 \leq p < \infty$, T_λ^k is a bounded operator from $L^p((0, \pi), w(\theta)dm_\lambda(\theta))$ into itself, when $1 < p < \infty$ and $w \in A_\lambda^p$, and from $L^1((0, \pi), w(\theta)dm_\lambda(\theta))$ into $L^{1,\infty}((0, \pi), w(\theta)dm_\lambda(\theta))$, when $w \in A_\lambda^1$. Hence, by [2, Theorem 1.4], we conclude that, for each $f \in L^p((0, \pi), w(\theta)dm_\lambda(\theta))$, $1 \leq p < \infty$,

$$R_\lambda^k(f)(\theta) = \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta-\varphi| > \varepsilon}^{\pi} R_\lambda^k(\theta, \varphi) f(\varphi) dm_\lambda(\varphi) + \gamma_k f(\theta), \quad \text{a.e. } \theta \in (0, \pi),$$

and the proof of this theorem would be finished.

Let now $f \in C_c^\infty(0, \pi)$ and $k \in \mathbb{N}$. We can write

$$L_\lambda^{-\frac{k}{2}} f(\theta) = \sum_{n=0}^{\infty} (n + \lambda)^{-k} a_n^\lambda(f) \frac{P_n^\lambda(\cos \theta)}{\|P_n^\lambda(\cos \cdot)\|_{L^2((0, \pi), dm_\lambda(\theta))}}, \quad \theta \in (0, \pi).$$

Then, since $f \in C_c^\infty(0, \pi)$, $L_\lambda^{-\frac{k}{2}} f \in C^\infty(0, \pi)$ (see [9, (2.4) and (2.6)]). We will see that

$$\frac{d^k}{d\theta^k} L_\lambda^{-\frac{k}{2}} f(\theta) = \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta-\varphi| > \varepsilon}^{\pi} f(\varphi) R_\lambda^k(\theta, \varphi) dm_\lambda(\varphi) + \gamma_k f(\theta), \quad \text{a.e. } \theta \in (0, \pi),$$

where

$$R_\lambda^k(\theta, \varphi) = \frac{\partial^k}{\partial \theta^k} \left(\frac{1}{\Gamma(k)} \int_0^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} P_\lambda(r, \theta, \varphi) dr \right), \quad \theta, \varphi \in (0, \pi),$$

$$P_\lambda(r, \theta, \varphi) = \frac{\lambda}{\pi} \int_0^\pi \frac{(1 - r^2)(\sin t)^{2\lambda-1}}{(1 - 2r(\cos \theta \cos \varphi + \sin \theta \sin \varphi \cos t) + r^2)^{\lambda+1}} dt, \quad r \in (0, 1), \theta, \varphi \in (0, \pi),$$

and $\gamma_k = 0$, when k is odd, and $\gamma_k = (-1)^{\frac{k}{2}}$, when k is even.

As in [2] we introduce the following useful notation:

$$\begin{aligned}\sigma &= \sin \theta \sin \varphi, \quad a = \cos \theta \cos \varphi + \sigma \cos t = \cos(\theta - \varphi) - \sigma(1 - \cos t), \\ b &= \frac{\partial}{\partial \theta} a = -\sin \theta \cos \varphi + \cos \theta \sin \varphi \cos t = -\sin(\theta - \varphi) - \cos \theta \sin \varphi(1 - \cos t), \\ \Delta_r &= 1 - 2r \cos(\theta - \varphi) + r^2 = (1 - r)^2 + 2r(1 - \cos(\theta - \varphi)), \quad \Delta = \Delta_1, \\ D_r &= 1 - 2ra + r^2 = \Delta_r + 2r\sigma(1 - \cos t).\end{aligned}$$

We divide the proof in several steps.

Step 1. We prove in the following that for every $\ell = 0, \dots, k-1$,

$$(4) \quad \frac{d^\ell}{d\theta^\ell} L_\lambda^{-\frac{k}{2}} f(\theta) = \int_0^\pi f(\varphi) R_\lambda^{k,\ell}(\theta, \varphi) dm_\lambda(\varphi), \quad \theta \in (0, \pi),$$

being

$$R_\lambda^{k,\ell}(\theta, \varphi) = \frac{1}{\Gamma(k)} \int_0^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} \frac{\partial^\ell}{\partial \theta^\ell} P_\lambda(r, \theta, \varphi) dr, \quad \theta, \varphi \in (0, \pi).$$

Let $\ell \in \mathbb{N}$, $0 \leq \ell \leq k-1$. Our objective is to establish that

$$(5) \quad \left| R_\lambda^{k,\ell}(\theta, \varphi) \right| \leq C \begin{cases} (\sin \varphi)^{-2\lambda-1}, & (\theta, \varphi) \in A_1; \\ \frac{1}{(\sin \theta \sin \varphi)^\lambda \sqrt{|\theta - \varphi|}}, & (\theta, \varphi) \in A_2, \theta \neq \varphi; \\ (\sin \theta)^{-2\lambda-1}, & (\theta, \varphi) \in A_3; \end{cases}$$

where $A_i, i = 1, 2, 3$, are the sets in the next figure:

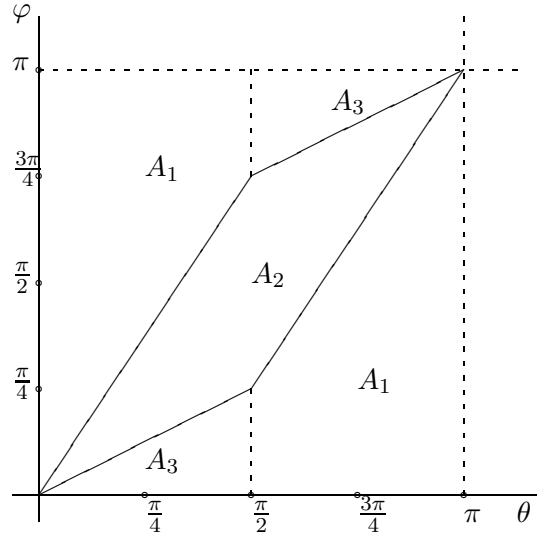


Figure 1

According to [2, Lemma 3.5] we have that

$$(6) \quad \frac{\partial^\ell}{\partial \theta^\ell} \left(\frac{1}{D_r^{\lambda+1}} \right) = \sum_{s,i,j} c_{\ell,s,i,j} \frac{r^{i+j} a^i b^j}{D_r^{\lambda+1+s}},$$

where $c_{\ell,s,i,j} \neq 0$ only if

$$(7) \quad s = 1, \dots, \ell, \quad j \geq 2s - \ell, \quad \text{and } i + j = s.$$

Moreover, by using the di Faà di Bruno's formula ([11, Theorem 2]), we can see that, for every $\ell \in \mathbb{N}$, $s = 1, \dots, \ell$, and $i + j = s$,

$$(8) \quad c_{\ell,s,i,j} = 2^s \ell! s! \sum \frac{(-1)^{s+j+\alpha(k_1, \dots, k_\ell)}}{k_1! \dots k_\ell! 1!^{k_1} 2!^{k_2} \dots \ell!^{k_\ell}},$$

where the sum is over all different solutions in nonnegative integers k_1, \dots, k_ℓ of the system

$$\left. \begin{aligned} k_1 + k_2 + \dots + k_\ell &= s \\ k_1 + 2k_2 + \dots + \ell k_\ell &= \ell \\ \sum_{r \text{ par}} k_r &= i \\ \sum_{r \text{ impar}} k_r &= j \end{aligned} \right\}$$

and

$$\alpha(k_1, \dots, k_\ell) = \sum_{r=2}^{\lfloor \frac{\ell}{2} \rfloor} (r-1)(k_{2r-1} + k_{2r}) + m_\ell,$$

being $m_\ell = 0$, if ℓ is even, and $m_\ell = \frac{(\ell-1)k_\ell}{2}$, when ℓ is odd.

We define, for every s, i, j satisfying (7),

$$M_{\ell,s,i,j}(\theta, \varphi) = \int_0^1 \int_0^\pi r^{i+j+\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{a^i b^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr, \quad \theta, \varphi \in (0, \pi).$$

In order to obtain (5) it is then sufficient to see that

$$(9) \quad |M_{\ell,s,i,j}(\theta, \varphi)| \leq C \begin{cases} (\sin \varphi)^{-2\lambda-1}, & (\theta, \varphi) \in A_1; \\ \frac{1}{(\sin \theta \sin \varphi)^\lambda \sqrt{|\theta - \varphi|}}, & (\theta, \varphi) \in A_2, \theta \neq \varphi; \\ (\sin \theta)^{-2\lambda-1}, & (\theta, \varphi) \in A_3; \end{cases}$$

for each s, i, j satisfying (7).

Moreover, by the symmetry of the Figure 1 and since

$$M_{\ell,s,i,j}(\pi - \theta, \pi - \varphi) = (-1)^j M_{\ell,s,i,j}(\theta, \varphi), \quad \theta, \varphi \in (0, \pi),$$

when s, i, j are as in (7), we can assume that $(\theta, \varphi) \in (0, \frac{\pi}{2}) \times (0, \pi)$.

Let us fix s, i, j verifying (7), $\theta \in (0, \frac{\pi}{2})$ and $\varphi \in (0, \pi)$. By proceeding as in [2, Lemma 3.6], and using that $\log \frac{1}{r} \sim 1 - r$, as $r \rightarrow 1^-$, and $D_r \geq C$, $r \in (0, \frac{1}{2})$, we get

$$\begin{aligned} |M_{\ell,s,i,j}(\theta, \varphi)| &\leq \left(\int_0^{\frac{1}{2}} \int_0^\pi + \int_{\frac{1}{2}}^1 \int_0^\pi \right) r^{i+j+\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{|a|^i |b|^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr \\ &\leq C \left(\int_0^{\frac{1}{2}} r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} dr + \int_{\frac{1}{2}}^1 \int_0^\pi (1-r)^k \frac{|b|^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr \right) \\ &\leq C \left(1 + \int_{\frac{1}{2}}^1 \int_0^\pi (1-r)^k \frac{|b|^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr \right). \end{aligned}$$

It can be seen that, if $(\alpha, \beta) \in [0, \pi] \times [0, \pi]$, $z \in (0, 1)$ and $\alpha \leq z\beta$, then there exists $C > 0$ such that $\sin(\beta - \alpha) \geq \min\{\sin \beta, \sin((1-z)\beta)\} \geq C \sin \beta$, and that, if $(\alpha, \beta) \in [0, \pi/2] \times [0, \pi]$ and $\frac{\alpha}{2} \leq \beta \leq \frac{3\alpha}{2}$, then $\sin |\beta - \alpha| \leq \sin \alpha$ and $\sin \alpha \sim \sin \beta$. These considerations allow us to write

$$(10) \quad |b|^j \leq C(|\sin(\theta - \varphi)|^j + (\sin \varphi)^j) \leq C \begin{cases} |\sin(\theta - \varphi)|^j, & \varphi \leq \frac{\theta}{2} \text{ or } \varphi \geq \frac{3\theta}{2}, \\ (\sin \varphi)^j, & \frac{\theta}{2} \leq \varphi \leq \frac{3\theta}{2}. \end{cases}$$

Then, since $1 - \cos \alpha \geq (\sin \alpha)^2 / \pi$, $\alpha \in [0, \pi]$, we obtain, when $\varphi \leq \frac{\theta}{2}$ or $\varphi \geq \frac{3\theta}{2}$,

$$\begin{aligned} |M_{\ell,s,i,j}(\theta, \varphi)| &\leq C \left(1 + (\sin |\theta - \varphi|)^j \int_{\frac{1}{2}}^1 \frac{(1-r)^k}{\Delta_r^{\lambda+s+1}} dr \right) \\ &\leq C \left(1 + (\sin |\theta - \varphi|)^j \int_{\frac{1}{2}}^1 \frac{(1-r)^{2s-j}}{(\Delta + (1-r)^2)^{\lambda+s+1}} dr \right) \\ &\leq C \left(1 + \frac{(\sin |\theta - \varphi|)^j}{\Delta^{\lambda+\frac{j+1}{2}}} \int_0^{\frac{1}{2\sqrt{\Delta}}} \frac{u^{2s-j}}{(1+u^2)^{\lambda+s+1}} du \right) \\ (11) \quad &\leq C \frac{1}{(\sin |\theta - \varphi|)^{2\lambda+1}} \\ &\leq C \begin{cases} (\sin \varphi)^{-2\lambda-1}, & \varphi \geq \frac{3\theta}{2}, \\ (\sin \theta)^{-2\lambda-1}, & \varphi \leq \frac{\theta}{2}. \end{cases} \end{aligned}$$

Suppose now that $\frac{\theta}{2} \leq \varphi \leq \frac{3\theta}{2}$, $\theta \neq \varphi$. One can write

$$\begin{aligned}
& \int_{\frac{1}{2}}^1 \int_0^\pi (1-r)^k \frac{|b|^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr \\
& \leq C \left(\int_{\frac{1}{2}}^1 \int_0^\pi (1-r)^k \frac{(\sin |\theta - \varphi|)^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr \right. \\
& \quad + \int_{\frac{1}{2}}^1 \int_0^{\frac{\pi}{2}} (1-r)^k \frac{(\sin \varphi (1 - \cos t))^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr \\
& \quad \left. + \int_{\frac{1}{2}}^1 \int_{\frac{\pi}{2}}^\pi (1-r)^k \frac{(\sin \varphi)^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr \right) = \sum_{\beta=1}^3 I_\beta(\theta, \varphi).
\end{aligned}$$

We analyze the first integral. By making two changes of variables, as in [2, p. 1235], and by taking into account that $2s - j \leq \ell \leq k - 1$ and that $\Delta = 2(1 - \cos(\theta - \varphi)) \sim (\sin |\theta - \varphi|)^2 \sim (\theta - \varphi)^2$ we get

$$\begin{aligned}
I_1(\theta, \varphi) & \leq C(\sin |\theta - \varphi|)^j \int_{\frac{1}{2}}^1 (1-r)^k \int_0^\pi \frac{(\sin t)^{2\lambda-1}}{(\Delta_r + 2r\sigma(1 - \cos t))^{\lambda+1+s}} dt dr \\
& \leq C(\sin |\theta - \varphi|)^j \int_{\frac{1}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{(\sin t)^{2\lambda-1}}{(\Delta_r + 2r\sigma(1 - \cos t))^{\lambda+1+s}} dt dr \\
& \leq C(\sin |\theta - \varphi|)^j \int_{\frac{1}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{t^{2\lambda-1}}{(\Delta_r + \sigma t^2)^{\lambda+1+s}} dt dr \\
& \leq C(\sin |\theta - \varphi|)^j \int_{\frac{1}{2}}^1 \frac{(1-r)^k}{\Delta_r^{\lambda+s+1}} \left(\sqrt{\frac{\Delta_r}{\sigma}} \right)^{2\lambda} \int_0^{\frac{\pi}{2} \sqrt{\frac{\sigma}{\Delta_r}}} \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+s+1}} du dr \\
& \leq C \frac{(\sin |\theta - \varphi|)^j}{\sigma^\lambda} \int_{\frac{1}{2}}^1 \frac{(1-r)^k}{\Delta_r^{1+s}} dr \\
& \leq C \frac{(\sin |\theta - \varphi|)^j}{\sigma^\lambda} \int_{\frac{1}{2}}^1 \frac{(1-r)^{2s-j+1/2}}{(\Delta + (1-r)^2)^{1+s}} dr \\
& \leq C \frac{(\sin |\theta - \varphi|)^j (\sqrt{\Delta})^{2s-j+3/2}}{\sigma^\lambda \Delta^{1+s}} \int_0^{\frac{1}{2\sqrt{\Delta}}} \frac{u^{2s-j+1/2}}{(1+u^2)^{1+s}} du \\
& \leq C \frac{(\sin |\theta - \varphi|)^j}{\sigma^\lambda \Delta^{\frac{1}{4} + \frac{j}{2}}} \int_0^\infty \frac{u^{2s-j+1/2}}{(1+u^2)^{1+s}} du \\
& \leq C \frac{1}{\sigma^\lambda \sqrt{|\theta - \varphi|}}.
\end{aligned}$$

For the second integral we write

$$\begin{aligned}
I_2(\theta, \varphi) &\leq C(\sin \varphi)^j \int_{\frac{1}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{t^{2\lambda+2j-1}}{(\Delta_r + \sigma t^2)^{\lambda+1+s}} dt dr \\
&\leq C(\sin \varphi)^j \int_{\frac{1}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{t^{2\lambda+j-1}}{(\Delta_r + \sigma t^2)^{\lambda+1+s}} dt dr \\
&\leq C \frac{(\sin \varphi)^j}{\sigma^{\lambda+\frac{j}{2}}} \int_{\frac{1}{2}}^1 \frac{(1-r)^k}{\Delta_r^{1+s-\frac{j}{2}}} dr \\
&\leq C \frac{(\sin \varphi)^j}{\sigma^{\lambda+\frac{j}{2}}} \int_{\frac{1}{2}}^1 \frac{(1-r)^{2s-j+\frac{1}{2}}}{(\Delta + (1-r)^2)^{1+s-\frac{j}{2}}} dr \\
&\leq C \frac{(\sin \varphi)^j (\sqrt{\Delta})^{2s-j+\frac{3}{2}}}{\sigma^{\lambda+\frac{j}{2}} \Delta^{1+s-\frac{j}{2}}} \int_0^{\frac{1}{2\sqrt{\Delta}}} \frac{u^{2s-j+\frac{1}{2}}}{(1+u^2)^{1+s-\frac{j}{2}}} du \\
&\leq C \frac{(\sin \varphi)^j}{\sigma^{\lambda+\frac{j}{2}} \Delta^{\frac{1}{4}}} \leq C \frac{1}{\sigma^\lambda \sqrt{|\theta - \varphi|}},
\end{aligned}$$

because $\sin \theta \sim \sin \varphi$. Finally it has

$$\begin{aligned}
I_3(\theta, \varphi) &\leq C(\sin \varphi)^j \int_{\frac{1}{2}}^1 \int_{\frac{\pi}{2}}^\pi (1-r)^k \frac{(\sin t)^{2\lambda-1}}{(\Delta_r + \sigma)^{\lambda+1+s}} dt dr \\
&\leq C \frac{(\sin \varphi)^j}{\sigma^{\lambda+\frac{j}{2}}} \int_{\frac{1}{2}}^1 \frac{(1-r)^k}{\Delta_r^{1+s-\frac{j}{2}}} dr \leq C \frac{1}{\sigma^\lambda \sqrt{|\theta - \varphi|}}.
\end{aligned}$$

Then we conclude that if $\frac{\theta}{2} \leq \varphi \leq \frac{3\theta}{2}$, $\theta \neq \varphi$,

$$|M_{\ell,s,i,j}(\theta, \varphi)| \leq C \left(1 + \int_{\frac{1}{2}}^1 \int_0^\pi (1-r)^k \frac{|b|^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+1+s}} dt dr \right) \leq C \frac{1}{\sigma^\lambda \sqrt{|\theta - \varphi|}},$$

that, jointly with (11), gives (9).

We have proved that the integral in (4) is absolutely convergent. By analyzing carefully the above estimates we can also see that, for every $\ell = 0, 1, \dots, k-2$, $R_\lambda^{k,\ell}$ is a continuous function on $(0, \pi) \times (0, \pi)$. Then, we conclude (4), for every $\ell = 0, 1, \dots, k-1$. Note that when $\ell = k-1$ to prove (4) we need to use distributional arguments (see Lemma 4.2 in Appendix).

Step 2. We now study the kernel

$$R_\lambda^k(\theta, \varphi) = \frac{\lambda}{\pi \Gamma(k)} \frac{\partial^k}{\partial \theta^k} \left[\int_0^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \int_0^\pi \frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}} dt dr \right],$$

for $\theta, \varphi \in (0, \pi)$. We get estimates which are better than the ones obtained in (5).

Lemma 2.1. *Let $\lambda > 0$ and $k \in \mathbb{N}$. Then,*

$$R_\lambda^k(\theta, \varphi) = \begin{cases} O((\sin \varphi)^{-(2\lambda+1)}), & (\theta, \varphi) \in A_1; \\ \frac{R^k(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} + O\left(\frac{1}{(\sin \varphi)^{2\lambda+1}} \left(1 + \sqrt{\frac{\sin \varphi}{|\theta - \varphi|}}\right)\right), & (\theta, \varphi) \in A_2, \theta \neq \varphi; \\ O((\sin \theta)^{-(2\lambda+1)}), & (\theta, \varphi) \in A_3; \end{cases}$$

where

$$R^k(\theta, \varphi) = \frac{1}{2\pi\Gamma(k)} \frac{\partial^k}{\partial \theta^k} \int_0^1 \left(\log \frac{1}{r}\right)^{k-1} \left(\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} - 1\right) \frac{dr}{r}.$$

Proof. Since $R_\lambda^k(\theta, \varphi) = (-1)^k R_\lambda^k(\pi - \theta, \pi - \varphi)$ and $R^k(\theta, \varphi) = (-1)^k R^k(\pi - \theta, \pi - \varphi)$, $\theta, \varphi \in (0, \pi)$, we can assume $(\theta, \varphi) \in (0, \frac{\pi}{2}) \times (0, \pi)$.

When $(\theta, \varphi) \in A_1 \cup A_3$, we can argue as in the proof of (11), for $\ell = k$, and thus we get

$$|R_\lambda^k(\theta, \varphi)| = \left| \frac{\lambda}{\pi\Gamma(k)} \sum_{\substack{s=1, \dots, k \\ j \geq 2s-k \\ i+j=s}} c_{k,s,i,j} M_{k,s,i,j}(\theta, \varphi) \right| \leq C \begin{cases} (\sin \varphi)^{-2\lambda-1}, & \varphi \geq \frac{3\theta}{2}, \\ (\sin \theta)^{-2\lambda-1}, & \varphi \leq \frac{\theta}{2}. \end{cases}$$

We now consider $\frac{\theta}{2} \leq \varphi \leq \frac{3\theta}{2}$ and $\varphi \neq \theta$. First we write

$$\begin{aligned} R_\lambda^k(\theta, \varphi) &= \frac{\lambda}{\pi\Gamma(k)} \left[\int_0^1 \int_{\frac{\pi}{2}}^\pi + \int_0^{1-\frac{\sqrt{\sigma}}{2}} \int_0^{\frac{\pi}{2}} + \int_{1-\frac{\sqrt{\sigma}}{2}}^1 \int_0^{\frac{\pi}{2}} \right] r^{\lambda-1} \left(\log \frac{1}{r}\right)^{k-1} (1-r^2) \\ &\quad \times \frac{\partial^k}{\partial \theta^k} \frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}} dt dr \\ (12) \quad &= \frac{\lambda}{\pi\Gamma(k)} [I_\lambda^{k,1}(\theta, \varphi) + I_\lambda^{k,2}(\theta, \varphi) + I_\lambda^{k,3}(\theta, \varphi)]. \end{aligned}$$

Let us decompose $I_\lambda^{k,3}(\theta, \varphi)$ as follows,

$$\begin{aligned} I_\lambda^{k,3}(\theta, \varphi) &= \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{\lambda-1} \left(\log \frac{1}{r}\right)^{k-1} (1-r^2) \frac{\partial^k}{\partial \theta^k} \int_0^{\frac{\pi}{2}} \left(\frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}} - \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} \right) dt dr \\ &\quad + \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{\lambda-1} \left(\log \frac{1}{r}\right)^{k-1} (1-r^2) \frac{\partial^k}{\partial \theta^k} \int_0^{\frac{\pi}{2}} \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} dt dr \\ (13) \quad &= J_\lambda^k(\theta, \varphi) + K_\lambda^k(\theta, \varphi). \end{aligned}$$

Moreover, we observe that by making the change of variable $u = \sqrt{\frac{r\sigma}{\Delta_r}}t$,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} dt &= \left(\int_0^\infty - \int_{\frac{\pi}{2}}^\infty \right) \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} dt \\ &= \frac{1}{(r\sigma)^\lambda \Delta_r} \int_0^\infty \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+1}} du - \int_{\frac{\pi}{2}}^\infty \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} dt \\ &= \frac{1}{2\lambda(r\sigma)^\lambda \Delta_r} - \int_{\frac{\pi}{2}}^\infty \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} dt, \quad r \in (0, 1). \end{aligned}$$

Then, Leibniz's rule leads to

$$\begin{aligned} K_\lambda^k(\theta, \varphi) &= \frac{1}{2\lambda} \left[\int_{1-\frac{\sqrt{\sigma}}{2}}^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{\partial^k}{\partial \theta^k} \left(\frac{1}{\sigma^\lambda} \frac{(1-r^2)}{\Delta_r} \right) \frac{dr}{r} \right] \\ &\quad - \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{\partial^k}{\partial \theta^k} \int_{\frac{\pi}{2}}^\infty \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} dt dr \\ &= \frac{1}{2\lambda\sigma^\lambda} \left[\int_{1-\frac{\sqrt{\sigma}}{2}}^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{\partial^k}{\partial \theta^k} \frac{(1-r^2)}{\Delta_r} \frac{dr}{r} \right] \\ &\quad + \frac{1}{2\lambda} \sum_{n=0}^{k-1} \binom{k}{n} \frac{\partial^{k-n}}{\partial \theta^{k-n}} \left(\frac{1}{\sigma^\lambda} \right) \left[\int_{1-\frac{\sqrt{\sigma}}{2}}^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{\partial^n}{\partial \theta^n} \frac{(1-r^2)}{\Delta_r} \frac{dr}{r} \right] \\ &\quad - \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{\partial^k}{\partial \theta^k} \int_{\frac{\pi}{2}}^\infty \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} dt dr. \end{aligned}$$

We observe that

$$(14) \quad K_\lambda^k(\theta, \varphi) = \frac{\pi \Gamma(k)}{\lambda \sigma^\lambda} R^k(\theta, \varphi) + \sum_{\beta=1}^3 K_\lambda^{k,\beta}(\theta, \varphi),$$

where

$$\begin{aligned} K_\lambda^{k,1}(\theta, \varphi) &= -\frac{1}{2\lambda\sigma^\lambda} \left[\int_0^{1-\frac{\sqrt{\sigma}}{2}} \left(\log \frac{1}{r} \right)^{k-1} \frac{\partial^k}{\partial \theta^k} \frac{(1-r^2)}{\Delta_r} \frac{dr}{r} \right], \\ K_\lambda^{k,2}(\theta, \varphi) &= \frac{1}{2\lambda} \sum_{n=0}^{k-1} \binom{k}{n} \frac{\partial^{k-n}}{\partial \theta^{k-n}} \left(\frac{1}{\sigma^\lambda} \right) \left[\int_{1-\frac{\sqrt{\sigma}}{2}}^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{\partial^n}{\partial \theta^n} \frac{(1-r^2)}{\Delta_r} \frac{dr}{r} \right], \end{aligned}$$

and

$$K_\lambda^{k,3}(\theta, \varphi) = - \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{\partial^k}{\partial \theta^k} \int_{\frac{\pi}{2}}^\infty \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}} dt dr.$$

Thus, according to (12), (13) and (14), to establish our result we must analyze $I_\lambda^{k,\beta}(\theta, \varphi)$, $\beta = 1, 2$, $J_\lambda^k(\theta, \varphi)$ and $K_\lambda^{k,\beta}(\theta, \varphi)$, $\beta = 1, 2, 3$.

Let us consider first $I_\lambda^{k,1}(\theta, \varphi)$. We will see that

$$(15) \quad |I_\lambda^{k,1}(\theta, \varphi)| \leq \frac{C}{(\sin \varphi)^{2\lambda+1}} \left(1 + \sqrt{\frac{\sin \varphi}{|\theta - \varphi|}} \right).$$

Let $s = 1, \dots, k$, $j \geq 2s - k$ and $i + j = s$. We define

$$I_{\lambda,s,i,j}^{k,1}(\theta, \varphi) = \int_0^1 r^{i+j+\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \int_{\frac{\pi}{2}}^\pi \frac{a^i b^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+s+1}} dt dr.$$

According to (6) it is then sufficient to obtain (15) when $I_{\lambda,s,i,j}^{k,1}(\theta, \varphi)$ replaces to $I_\lambda^{k,1}(\theta, \varphi)$. By proceeding as in Step 1, using (10), since $D_r \geq C$, for $0 < r < \frac{1}{2}$, and $D_r \geq (\Delta_r + \sigma)$, for $\frac{1}{2} < r < 1$ and $t \in (\frac{\pi}{2}, \pi)$, it has

$$\begin{aligned} |I_{\lambda,s,i,j}^{k,1}(\theta, \varphi)| &\leq C \int_0^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \int_{\frac{\pi}{2}}^\pi \frac{|b|^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+s+1}} dt dr \\ &\leq C \left(1 + (\sin \varphi)^j \int_{\frac{1}{2}}^1 \frac{(1-r)^k}{(\Delta_r + \sigma)^{\lambda+s+1}} dr \right) \\ &\leq C \left(1 + \frac{(\sin \varphi)^j}{\sigma^{\lambda+\frac{1}{4}+\frac{j}{2}}} \int_{\frac{1}{2}}^1 \frac{(1-r)^{2s-j}}{\Delta_r^{s+\frac{3}{4}-\frac{j}{2}}} dr \right) \\ &\leq C \left(1 + \frac{1}{\sigma^{\lambda+\frac{1}{4}} \Delta^{\frac{1}{4}}} \int_0^\infty \frac{u^{2s-j}}{(1+u^2)^{s+\frac{3}{4}-\frac{j}{2}}} du \right) \\ &\leq C \left(1 + \frac{1}{(\sin \varphi)^{2\lambda+\frac{1}{2}} \sqrt{|\theta - \varphi|}} \right) \leq \frac{C}{(\sin \varphi)^{2\lambda+1}} \left(1 + \sqrt{\frac{\sin \varphi}{|\theta - \varphi|}} \right). \end{aligned}$$

For $I_\lambda^{k,2}(\theta, \varphi)$ we proceed in a similar way. Consider $s = 1, \dots, k$, $j \geq 2s - k$, and $i + j = s$, and

$$I_{\lambda,s,i,j}^{k,2}(\theta, \varphi) = \int_0^{1-\frac{\sqrt{\sigma}}{2}} r^{i+j+\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \int_0^{\frac{\pi}{2}} \frac{a^i b^j (\sin t)^{2\lambda-1}}{D_r^{\lambda+s+1}} dt dr.$$

We have that

$$\begin{aligned} |I_{\lambda,s,i,j}^{k,2}(\theta, \varphi)| &\leq C \left(\int_0^{\frac{1}{2}} r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} dr + (\sin \varphi)^j \int_{\frac{1}{2}}^{1-\frac{\sqrt{\sigma}}{2}} \frac{(1-r)^k}{\Delta_r^{\lambda+s+1}} dr \right) \\ &\leq C \left(1 + (\sin \varphi)^j \int_{\frac{1}{2}}^{1-\frac{\sqrt{\sigma}}{2}} \frac{(1-r)^{2s-j}}{(1-r)^{2\lambda+2s+2}} dr \right) \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}. \end{aligned}$$

Hence,

$$(16) \quad |I_\lambda^{k,2}(\theta, \varphi)| \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}.$$

To estimate $J_\lambda^k(\theta, \varphi)$ (see (13)), we write

$$J_\lambda^k(\theta, \varphi) = J_\lambda^{k,1}(\theta, \varphi) + J_\lambda^{k,2}(\theta, \varphi),$$

where

$$J_\lambda^{k,1}(\theta, \varphi) = \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{\partial^k}{\partial \theta^k} \int_0^{\frac{\pi}{2}} \frac{(\sin t)^{2\lambda-1} - t^{2\lambda-1}}{D_r^{\lambda+1}} dt dr,$$

and

$$J_\lambda^{k,2}(\theta, \varphi) = \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{\partial^k}{\partial \theta^k} \int_0^{\frac{\pi}{2}} t^{2\lambda-1} \left(\frac{1}{D_r^{\lambda+1}} - \frac{1}{(\Delta_r + r\sigma t^2)^{\lambda+1}} \right) dt dr.$$

To analyze $J_\lambda^{k,1}(\theta, \varphi)$ assume, as above, $s = 1, \dots, k$, $j \geq 2s - k$, and $i + j = s$ and consider

$$J_{\lambda,s,i,j}^{k,1}(\theta, \varphi) = \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{i+j+\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \int_0^{\frac{\pi}{2}} \frac{a^i b^j [(\sin t)^{2\lambda-1} - t^{2\lambda-1}]}{D_r^{\lambda+s+1}} dt dr.$$

By using the mean value theorem and that $|b|^j \leq C(|\theta - \varphi|^j + (t^2 \sin \varphi)^j)$, $t \in (0, \frac{\pi}{2})$, we have

$$\begin{aligned} (17) \quad |J_{\lambda,s,i,j}^{k,1}(\theta, \varphi)| &\leq C \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{|b|^j t^{2\lambda+1}}{D_r^{\lambda+s+1}} dt dr \\ &\leq C \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{[|\theta - \varphi|^j + (t^2 \sin \varphi)^j] t^{2\lambda+1}}{(\Delta_r + \sigma t^2)^{\lambda+s+1}} dt dr \\ &= C(J_{\lambda,s,i,j}^{k,1,1}(\theta, \varphi) + J_{\lambda,s,i,j}^{k,1,2}(\theta, \varphi)). \end{aligned}$$

We can obtain for each term in the last sum the following estimates. Firstly,

$$\begin{aligned} |J_{\lambda,s,i,j}^{k,1,1}(\theta, \varphi)| &\leq C \frac{|\theta - \varphi|^j}{\sigma^{\lambda+1}} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 \frac{(1-r)^k}{\Delta_r^s} \int_0^{\frac{\pi}{2} \sqrt{\frac{\sigma}{\Delta_r}}} \frac{u^{2\lambda+1}}{(1+u^2)^{\lambda+s+1}} du dr \\ &\leq C \frac{|\theta - \varphi|^j}{\sigma^{\lambda+1}} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 \frac{(1-r)^k}{(\Delta + (1-r)^2)^{s-\frac{j}{2}+\frac{j}{2}}} dr \\ &\leq C \frac{1}{\sigma^{\lambda+1}} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^{k-2s+j} dr \leq C \frac{\sigma^{\frac{k+j}{2}-s}}{\sigma^{\lambda+\frac{1}{2}}} \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}. \end{aligned}$$

We also have

$$\begin{aligned}
|J_{\lambda,s,i,j}^{k,1,2}(\theta, \varphi)| &\leq C(\sin \varphi)^j \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{t^{2\lambda+2j+1}}{((1-r)^2 + \sigma t^2)^{\lambda+s+1}} dt dr \\
&\leq C \frac{(\sin \varphi)^j}{\sigma^{\lambda+\frac{j}{2}+\frac{3}{4}}} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^{k-2s+j-\frac{1}{2}} dr \int_0^{\frac{\pi}{2}} t^{j-\frac{1}{2}} dt \\
&\leq C \frac{\sigma^{\frac{k+j}{2}-s}}{\sigma^{\lambda+\frac{1}{2}}} \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}.
\end{aligned}$$

Thus we get by (6)

$$(18) \quad |J_{\lambda}^{k,1}(\theta, \varphi)| \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}.$$

In the same way, and according to (6), to see the estimate for $J_{\lambda}^{k,2}(\theta, \varphi)$ we analyze

$$\begin{aligned}
J_{\lambda,s,i,j}^{k,2}(\theta, \varphi) &= \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{\lambda+i+j-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \\
&\quad \times \int_0^{\frac{\pi}{2}} t^{2\lambda-1} \left(\frac{a^i b^j}{D_r^{\lambda+s+1}} - \frac{A^i B^j}{(\Delta_r + r\sigma t^2)^{\lambda+s+1}} \right) dt dr,
\end{aligned}$$

for every $s = 1, \dots, k$, $j \geq 2s - k$, and $i + j = s$. Here $A = \cos(\theta - \varphi) - \frac{\sigma t^2}{2}$ and $B = \frac{\partial A}{\partial \theta}$.

By using the mean value theorem we obtain, for $t \in (0, \frac{\pi}{2})$,

$$\begin{aligned}
&\left| \frac{a^i b^j}{D_r^{\lambda+s+1}} - \frac{A^i B^j}{(\Delta_r + r\sigma t^2)^{\lambda+s+1}} \right| \\
&\leq \left| a^i b^j \left(\frac{1}{D_r^{\lambda+s+1}} - \frac{1}{(\Delta_r + r\sigma t^2)^{\lambda+s+1}} \right) \right| + \left| \frac{(a^i - A^i) b^j + A^i (b^j - B^j)}{(\Delta_r + r\sigma t^2)^{\lambda+s+1}} \right| \\
&\leq C \left(\frac{|b|^j \sigma t^4}{(\Delta_r + r\sigma t^2)^{\lambda+s+2}} + \frac{|b|^{j-1} \sqrt{\sigma} t^4}{(\Delta_r + r\sigma t^2)^{\lambda+s+1}} \right) \\
&\leq C \left(\frac{|b|^j t^2}{(\Delta_r + \sigma t^2)^{\lambda+s+1}} + \frac{|b|^{j-1} \sqrt{\sigma} t^3}{(\Delta_r + \sigma t^2)^{\lambda+s+1}} \right),
\end{aligned}$$

where the second term in the two last sums does not appear when $j = 0$. Then, we write

$$|J_{\lambda,s,i,j}^{k,2}(\theta, \varphi)| \leq C(J_{\lambda,s,i,j}^{k,2,1}(\theta, \varphi) + J_{\lambda,s,i,j}^{k,2,2}(\theta, \varphi)),$$

where

$$J_{\lambda,s,i,j}^{k,2,1}(\theta, \varphi) = \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{|b|^j t^{2\lambda+1}}{(\Delta_r + \sigma t^2)^{\lambda+s+1}} dt dr,$$

and

$$J_{\lambda,s,i,j}^{k,2,2}(\theta, \varphi) = \sqrt{\sigma} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^k \int_0^{\frac{\pi}{2}} \frac{|b|^{j-1} t^{2\lambda+2}}{(\Delta_r + \sigma t^2)^{\lambda+s+1}} dt dr, \quad \text{when } j \geq 1.$$

We observe that $J_{\lambda,s,i,j}^{k,2,1}(\theta, \varphi)$ was already analyzed in (17). On the other hand, when $j \geq 1$, we can use that $|b|^{j-1} \leq C(|\theta - \varphi|^{j-1} + (t^2 \sin \varphi)^{j-1})$, $t \in (0, \frac{\pi}{2})$, and proceed as in the estimation of (17) to study $J_{\lambda,s,i,j}^{k,2,2}$. Thus we get that

$$(19) \quad |J_{\lambda}^{k,2}(\theta, \varphi)| \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}.$$

By combining (18) and (19) we conclude that

$$(20) \quad |J_{\lambda}^k(\theta, \varphi)| \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}.$$

Finally we deal with $K_{\lambda}^{k,\beta}(\theta, \varphi)$, $\beta = 1, 2, 3$ (see (14)).

By invoking (6) with $t = 0$ and $\lambda = 0$, it has that

$$\frac{\partial^{\ell}}{\partial \theta^{\ell}} \left(\frac{1}{\Delta_r} \right) = \sum_{s,i,j} c_{\ell,s,i,j} \frac{r^{i+j} (\cos(\theta - \varphi))^i (-\sin(\theta - \varphi))^j}{\Delta_r^{1+s}}, \quad \ell \in \mathbb{N},$$

where $c_{\ell,s,i,j} \neq 0$ only if $s = 1, \dots, \ell$, $j \geq 2s - \ell$ and $i + j = s$.

Also, for every $n = 0, \dots, k - 1$,

$$(21) \quad \left| \frac{\partial^{k-n}}{\partial \theta^{k-n}} \left(\frac{1}{\sigma^{\lambda}} \right) \right| \leq C (\sin \theta)^{-\lambda-k+n} (\sin \varphi)^{-\lambda} \leq C \sigma^{-\lambda - \frac{k-n}{2}}.$$

Hence, by proceeding as above, to estimate $K_{\lambda}^{k,\beta}(\theta, \varphi)$, $\beta = 1, 2, 3$, it is sufficient to study the following integrals:

$$K_{\lambda,s,i,j}^{k,1}(\theta, \varphi) = \frac{1}{\sigma^{\lambda}} \int_0^{1-\frac{\sqrt{\sigma}}{2}} r^{i+j-1} \left(\log \frac{1}{r} \right)^{k-1} \frac{(1-r^2)(\cos(\theta - \varphi))^i (-\sin(\theta - \varphi))^j}{\Delta_r^{s+1}} dr,$$

when $s = 1, \dots, k$, $j \geq 2s - k$ and $i + j = s$;

$$K_{\lambda}^{k,2,0}(\theta, \varphi) = \sigma^{-k/2-\lambda} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{(1-r^2)}{\Delta_r} \frac{dr}{r};$$

$$K_{\lambda,s,i,j}^{k,2,n}(\theta, \varphi) = \frac{\partial^{k-n}}{\partial \theta^{k-n}} \left(\frac{1}{\sigma^{\lambda}} \right) \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{i+j-1} \left(\log \frac{1}{r} \right)^{k-1} \frac{(1-r^2)(\cos(\theta - \varphi))^i (-\sin(\theta - \varphi))^j}{\Delta_r^{s+1}} dr,$$

for each $n = 1, \dots, k - 1$, $s = 1, \dots, n$, $j \geq 2s - n$, and $i + j = s$; and

$$K_{\lambda,s,i,j}^{k,3}(\theta, \varphi) = \int_{1-\frac{\sqrt{\sigma}}{2}}^1 r^{i+j+\lambda-1} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \int_{\frac{\pi}{2}}^{\infty} \frac{A^i B^j t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+s+1}} dt dr,$$

when $s = 1, \dots, k$, $j \geq 2s - k$, and $i + j = s$. Here, as before, $A = \cos(\theta - \varphi) - \frac{\sigma t^2}{2}$ and $B = \frac{\partial A}{\partial \theta}$.

Consider $s = 1, \dots, k$, $j \geq 2s - k$, and $i + j = s$. We can write

$$\begin{aligned}
|K_{\lambda,s,i,j}^{k,1}(\theta, \varphi)| &\leq C \frac{1}{\sigma^\lambda} \left(\int_0^{\frac{1}{2}} r^{i+j-1} \left(\log \frac{1}{r} \right)^{k-1} dr + \int_{\frac{1}{2}}^{1-\frac{\sqrt{\sigma}}{2}} \frac{(1-r)^k |\theta - \varphi|^j}{(\Delta + (1-r)^2)^{s+1}} dr \right) \\
&\leq C \frac{1}{\sigma^\lambda} \left(1 + \frac{|\theta - \varphi|^j}{\Delta^{\frac{j}{2}}} \int_{\frac{1}{2}}^{1-\frac{\sqrt{\sigma}}{2}} (1-r)^{k-2s+j-2} dr \right) \\
&\leq C \frac{1}{\sigma^\lambda} \left(1 + \frac{\sigma^{\frac{k+j}{2}-s}}{\sqrt{\sigma}} \right) \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}.
\end{aligned}$$

On the other hand, by using (21), for each $n = 1, \dots, k-1$, $s = 1, \dots, n$, $j \geq 2s - n$, and $i + j = s$, we obtain

$$\begin{aligned}
|K_{\lambda,s,i,j}^{k,2,n}(\theta, \varphi)| &\leq C \frac{|\theta - \varphi|^j}{\sigma^{\lambda+\frac{k-n}{2}}} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 \frac{(1-r)^k}{(\Delta + (1-r)^2)^{s+1}} dr \\
&\leq C \frac{|\theta - \varphi|^j}{\sigma^{\lambda+\frac{k-n}{2}} \Delta^{\frac{j}{2}+\frac{1}{4}}} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^{k-2s+j-\frac{3}{2}} dr \\
&\leq C \frac{\sigma^{\frac{n+j}{2}-s}}{\sigma^{\lambda+\frac{1}{4}} \Delta^{\frac{1}{4}}} \leq \frac{C}{(\sin \varphi)^{2\lambda+1}} \sqrt{\frac{\sin \varphi}{|\theta - \varphi|}}.
\end{aligned}$$

In a similar way we obtain

$$|K_{\lambda}^{k,2,0}(\theta, \varphi)| \leq \frac{C}{(\sin \varphi)^{2\lambda+1}} \sqrt{\frac{\sin \varphi}{|\theta - \varphi|}}.$$

Finally, assume $s = 1, \dots, k$, $j \geq 2s - k$, and $i + j = s$. By taking into account that $|B|^j \leq C(t^2 \sin \varphi)^j$ and $|A|^i \leq C(1 + \sigma^i t^{2i})$, $t \geq \frac{\pi}{2}$, and the last formula in [10, p. 37], we obtain

$$\begin{aligned}
K_{\lambda,s,i,j}^{k,3}(\theta, \varphi) &\leq C \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^k \int_{\frac{\pi}{2}}^{\infty} \frac{|A|^i |B|^j t^{2\lambda-1}}{(\Delta_r + \sigma t^2)^{\lambda+s+1}} dt dr \\
&\leq C \frac{(\sin \varphi)^j}{\sigma^{\lambda+j}} \left(\int_{1-\frac{\sqrt{\sigma}}{2}}^1 \frac{(1-r)^k}{\Delta_r^{s+1-j}} \int_{\frac{\pi}{2}\sqrt{\frac{\sigma}{\Delta_r}}}^{\infty} \frac{u^{2\lambda+2j-1}}{(1+u^2)^{\lambda+s+1}} du dr \right. \\
&\quad \left. + \int_{1-\frac{\sqrt{\sigma}}{2}}^1 \frac{(1-r)^k}{\Delta_r} \int_{\frac{\pi}{2}\sqrt{\frac{\sigma}{\Delta_r}}}^{\infty} \frac{u^{2\lambda+2j+2i-1}}{(1+u^2)^{\lambda+s+1}} du dr \right) \\
&\leq C \frac{(\sin \varphi)^j}{\sigma^{\lambda+j+1}} \left(\int_{1-\frac{\sqrt{\sigma}}{2}}^1 \frac{(1-r)^k}{(\Delta + (1-r)^2)^{s-j}} dr + \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^k dr \right) \\
&\leq C \frac{(\sin \varphi)^j}{\sigma^{\lambda+j+1}} \int_{1-\frac{\sqrt{\sigma}}{2}}^1 (1-r)^{k-2s+2j} dr \leq C \frac{(\sin \varphi)^j \sigma^{\frac{k+j}{2}-s}}{\sigma^{\lambda+\frac{j}{2}+\frac{1}{2}}} \leq \frac{C}{(\sin \varphi)^{2\lambda+1}}.
\end{aligned}$$

Thus, we have obtained that

$$(22) \quad \sum_{\beta=1}^3 |K_{\lambda}^{k,\beta}(\theta, \varphi)| \leq \frac{C}{(\sin \varphi)^{2\lambda+1}} \left(1 + \sqrt{\frac{\sin \varphi}{|\theta - \varphi|}} \right).$$

By considering (12), (13), (14) and the estimations (15), (16), (20) and (22) we conclude that, when $\frac{\theta}{2} \leq \varphi \leq \frac{3\theta}{2}$, $\theta \neq \varphi$,

$$\left| R_{\lambda}^k(\theta, \varphi) - \frac{R^k(\theta, \varphi)}{\sigma^{\lambda}} \right| \leq \frac{C}{(\sin \varphi)^{2\lambda+1}} \left(1 + \sqrt{\frac{\sin \varphi}{|\theta - \varphi|}} \right),$$

and the proof of Lemma 2.1 is finished. \square

Step 3. We now establish that the k -th Riesz transform in the circle is a principal value integral operator, that is,

$$(23) \quad \begin{aligned} \frac{d^k}{d\theta^k} \int_0^{\pi} f(\varphi) \int_0^1 \left(\log \frac{1}{r} \right)^{k-1} \left(\frac{1-r^2}{1-2r \cos(\theta-\varphi)+r^2} - 1 \right) \frac{dr}{r} d\varphi \\ = 2\pi \Gamma(k) \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta-\varphi| > \varepsilon}^{\pi} f(\varphi) R^k(\theta, \varphi) d\varphi + \beta_k f(\theta), \end{aligned}$$

for every $\theta \in (0, \pi)$, and where $\beta_k = 0$ when k is odd, and $\beta_k = 2\pi(-1)^{\frac{k}{2}}\Gamma(k)$, when k is even.

Let us consider the function

$$H^k(\omega) = \int_0^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{\partial^{k-1}}{\partial \omega^{k-1}} \left(\frac{1-r^2}{1-2r \cos \omega + r^2} - 1 \right) \frac{dr}{r}, \quad \omega \in \mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}.$$

Firstly we are going to analyze the behavior of $H^k(w)$ when $w \rightarrow 0^+$. We have that

$$\begin{aligned} H^1(w) &= \int_0^1 \left(\frac{1-r^2}{1-2r \cos w + r^2} - 1 \right) \frac{dr}{r} \\ &= -\log(2(1 - \cos w)), \quad w \in \mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}. \end{aligned}$$

Then, $\lim_{w \rightarrow 0} w H^1(w) = 0$.

Assume that $k \in \mathbb{N}$, $k \geq 2$. According to (6), it has

$$\begin{aligned} H^k(w) &= \sum_{\substack{s=1, \dots, k-1 \\ j \geq 2s-k+1 \\ i+j=s}} c_{k-1,s,i,j} \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1 \right) \frac{r^{i+j} (\cos w)^i (-\sin w)^j}{(1-2r \cos w + r^2)^{s+1}} (1-r^2) \left(\log \frac{1}{r} \right)^{k-1} \frac{dr}{r} \\ &= \sum_{\substack{s=1, \dots, k-1 \\ j \geq 2s-k+1 \\ i+j=s}} c_{k-1,s,i,j} (I_{k,s,i,j}^0(w) + I_{k,s,i,j}^1(w)), \quad w \in \mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}. \end{aligned}$$

Let $s = 1, \dots, k-1$, $i+j = s$, $j \geq 2s-k+1$. By using the dominated convergence theorem we obtain

$$\lim_{w \rightarrow 0} I_{k,s,i,j}^0(w) = \begin{cases} 0, & j \geq 1; \\ \int_0^{1/2} \left(\log \frac{1}{r} \right)^{k-1} \frac{(1-r^2)r^{s-1}}{(1-r)^{2(s+1)}} dr, & j = 0; \end{cases}$$

and, when $2s-k+1 < 0$,

$$\lim_{w \rightarrow 0} I_{k,s,i,j}^1(w) = \begin{cases} 0, & j \geq 1; \\ \int_{1/2}^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{(1-r^2)r^{s-1}}{(1-r)^{2(s+1)}} dr, & j = 0. \end{cases}$$

Assume now $2s-k+1 \geq 0$. We have that

$$|I_{k,s,i,j}^1(w)| \leq C|w|^{j+k-2s-1} \int_0^{\frac{1}{2|w|}} \frac{u^k}{(1+u^2)^{s+1}} du \leq C|w|^{j+k-2s-\frac{3}{2}}, \quad w \in (-\pi, \pi) \setminus \{0\}.$$

Then,

$$\lim_{w \rightarrow 0} w I_{k,s,i,j}^1(w) = 0,$$

and if $j > 2s-k+1$,

$$\lim_{w \rightarrow 0} I_{k,s,i,j}^1(w) = 0.$$

Also, if $j = 2s-k+1 > 0$ (as will be the case if k is even), by using mean value theorem it follows

$$\begin{aligned} \lim_{w \rightarrow 0^+} I_{k,s,i,j}^1(w) &= -2 \lim_{w \rightarrow 0^+} \left(\frac{\sin w}{w} \right)^{2s-k+1} \int_0^{\frac{1}{2w}} \frac{u^k}{(1+u^2)^{s+1}} du \\ &= -2 \int_0^\infty \frac{u^k}{(1+u^2)^{s+1}} du = -B\left(\frac{k+1}{2}, s - \frac{k-1}{2}\right), \end{aligned}$$

where $B(x, y)$, $x, y > 0$, represents the Beta Euler's function.

By combining the above estimates we conclude that $\lim_{w \rightarrow 0} w H^k(w) = 0$, when k is odd. Assume now that k is even. In this case we obtain that

$$\begin{aligned} \lim_{w \rightarrow 0^+} H^k(w) &= \sum_{s=1}^{\frac{k}{2}-1} c_{k-1,s,s,0} \int_0^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{(1-r^2)r^{s-1}}{(1-r)^{2(s+1)}} dr \\ &\quad - \sum_{s=\frac{k}{2}}^{k-1} c_{k-1,s,k-s-1,2s-k+1} B\left(\frac{k+1}{2}, s - \frac{k-1}{2}\right). \end{aligned}$$

By taking into account (8) and the duplication formula for the Gamma Euler's function, we can write

$$\begin{aligned}
\lim_{w \rightarrow 0^+} H^k(w) &= - \sum_{s=\frac{k}{2}}^{k-1} \frac{(-1)^{s+1} (k-1)! s!}{2^{k-2s-1} (2s-k+1)! (k-s-1)!} B\left(\frac{k+1}{2}, s - \frac{k-1}{2}\right) \\
&= \frac{\pi(\Gamma(k))^2}{2^{k-2}\Gamma(\frac{k}{2})} \sum_{s=\frac{k}{2}}^{k-1} \frac{(-1)^s}{(2s-k+1)(k-s-1)!(s-\frac{k}{2})!} = \frac{(-1)^{\frac{k}{2}} \pi(\Gamma(k))^2}{2^{k-2}(\Gamma(\frac{k}{2}))^2} \sum_{r=0}^{\frac{k}{2}-1} (-1)^r \binom{\frac{k}{2}-1}{r} \frac{1}{2r+1} \\
&= \frac{(-1)^{\frac{k}{2}} \pi(\Gamma(k))^2}{2^{k-2}(\Gamma(\frac{k}{2}))^2} \int_0^1 (1-t^2)^{\frac{k}{2}-1} dt = (-1)^{\frac{k}{2}} \pi \Gamma(k).
\end{aligned}$$

By proceeding as in Step 1 we can see that

$$\begin{aligned}
&\frac{d^{k-1}}{d\theta^{k-1}} \int_0^\pi f(\varphi) \int_0^1 \left(\log \frac{1}{r}\right)^{k-1} \left(\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} - 1\right) \frac{dr}{r} d\varphi \\
(24) \quad &= \int_0^\pi f(\varphi) \int_0^1 \left(\log \frac{1}{r}\right)^{k-1} \frac{\partial^{k-1}}{\partial \theta^{k-1}} \left(\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} - 1\right) \frac{dr}{r} d\varphi, \quad \theta \in (0, \pi).
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\frac{d^{k-1}}{d\theta^{k-1}} \int_0^\pi f(\varphi) \int_0^1 \left(\log \frac{1}{r}\right)^{k-1} \left(\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} - 1\right) \frac{dr}{r} d\varphi \\
&= \int_{-\pi}^\pi \tilde{f}(\varphi) H^k(\theta - \varphi) d\varphi, \quad \theta \in (0, \pi),
\end{aligned}$$

$$\text{where } \tilde{f}(\varphi) = \begin{cases} f(\varphi), & \varphi \in [0, \pi) \\ 0, & \varphi \in (-\pi, 0) \end{cases}, \text{ and } \tilde{f}(\varphi) = \tilde{f}(\varphi + 2\pi), \varphi \in \mathbb{R}.$$

Also, since $H^k \in L^1(-\pi, \pi)$, we have that,

$$\begin{aligned}
&\frac{d}{d\theta} \int_{-\pi}^\pi \tilde{f}(\varphi) H^k(\theta - \varphi) d\varphi = \frac{\partial}{\partial \theta} \int_{-\pi}^\pi \tilde{f}(\theta - u) H^k(u) du = \int_{-\pi}^\pi \frac{\partial}{\partial \theta} \tilde{f}(\theta - u) H^k(u) du \\
&= - \lim_{\varepsilon \rightarrow 0^+} \int_{-\pi, |u| > \varepsilon}^\pi \frac{d}{du} [\tilde{f}(\theta - u)] H^k(u) du \\
&= - \lim_{\varepsilon \rightarrow 0^+} \left(\tilde{f}(\theta - u) H^k(u) \Big|_\varepsilon^\pi - \int_\varepsilon^\pi \tilde{f}(\theta - u) \frac{d}{du} H^k(u) du \right. \\
&\quad \left. + \tilde{f}(\theta - u) H^k(u) \Big|_{-\pi}^{-\varepsilon} - \int_{-\pi}^{-\varepsilon} \tilde{f}(\theta - u) \frac{d}{du} H^k(u) du \right) \\
&= - \lim_{\varepsilon \rightarrow 0^+} \left(\tilde{f}(\theta - \pi) H^k(\pi) - \tilde{f}(\theta - \varepsilon) H^k(\varepsilon) + \tilde{f}(\theta + \varepsilon) H^k(-\varepsilon) \right. \\
&\quad \left. - \tilde{f}(\theta + \pi) H^k(-\pi) - \int_{-\pi, |u| > \varepsilon}^\pi \tilde{f}(\theta - u) \frac{d}{du} H^k(u) du \right), \quad \theta \in (0, \pi).
\end{aligned}$$

Since the function H^k is even when k is odd and H^k is odd when k is even, we conclude that

$$\begin{aligned} \frac{d}{d\theta} \int_0^\pi f(\varphi) H^k(\theta - \varphi) d\varphi &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\pi, |\theta - \varphi| > \varepsilon}^\pi \tilde{f}(\varphi) \left(\frac{d}{du} H^k \right) (\theta - \varphi) d\varphi \\ &- \lim_{\varepsilon \rightarrow 0^+} (f(\theta + \varepsilon) - f(\theta - \varepsilon)) H^k(\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta - \varphi| > \varepsilon}^\pi f(\varphi) \frac{\partial}{\partial \theta} H^k(\theta - \varphi) d\varphi, \quad \theta \in (0, \pi), \end{aligned}$$

when k is odd, and

$$\begin{aligned} \frac{d}{d\theta} \int_0^\pi f(\varphi) H^k(\theta - \varphi) d\varphi &= \lim_{\varepsilon \rightarrow 0^+} (f(\theta + \varepsilon) + f(\theta - \varepsilon)) H^k(\varepsilon) + \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta - \varphi| > \varepsilon}^\pi f(\varphi) \frac{\partial}{\partial \theta} H^k(\theta - \varphi) d\varphi \\ &= 2f(\theta)(-1)^{\frac{k}{2}} \pi \Gamma(k) + \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta - \varphi| > \varepsilon}^\pi f(\varphi) \frac{\partial}{\partial \theta} H^k(\theta - \varphi) d\varphi, \quad \theta \in (0, \pi), \end{aligned}$$

when k is even.

Step 4. We now finish the proof of Theorem 1.1. We firstly write, according to (4),

$$\begin{aligned} \frac{d^{k-1}}{d\theta^{k-1}} L_\lambda^{-\frac{k}{2}} f(\theta) &= \int_0^\pi f(\varphi) R_\lambda^{k,k-1}(\theta, \varphi) dm_\lambda(\varphi) = \int_0^\pi f(\varphi) \frac{R^{k,k-1}(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} dm_\lambda(\varphi) \\ &+ \int_0^\pi f(\varphi) \left(R_\lambda^{k,k-1}(\theta, \varphi) - \frac{R^{k,k-1}(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} \right) dm_\lambda(\varphi), \quad \theta \in (0, \pi), \end{aligned}$$

where, for every $\theta, \varphi \in (0, \pi)$,

$$R^{k,k-1}(\theta, \varphi) = \frac{1}{2\pi\Gamma(k)} \int_0^1 \left(\log \frac{1}{r} \right)^{k-1} \frac{\partial^{k-1}}{\partial \theta^{k-1}} \left(\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} - 1 \right) \frac{dr}{r}.$$

Moreover, by (23) and (24) we have, for every $\theta \in (0, \pi)$,

$$\begin{aligned} (25) \quad \frac{d}{d\theta} \int_0^\pi f(\varphi) \frac{R^{k,k-1}(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} dm_\lambda(\varphi) &= \frac{1}{2\pi\Gamma(k)} \frac{d}{d\theta} \left(\frac{1}{(\sin \theta)^\lambda} \right. \\ &\times \left. \frac{d^{k-1}}{d\theta^{k-1}} \int_0^\pi \frac{f(\varphi)}{(\sin \varphi)^\lambda} \int_0^1 \left(\log \frac{1}{r} \right)^{k-1} \left(\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} - 1 \right) \frac{dr}{r} dm_\lambda(\varphi) \right) \\ &= -\frac{\lambda \cos \theta}{(\sin \theta)^{\lambda+1}} \int_0^\pi \frac{f(\varphi)}{(\sin \varphi)^\lambda} R^{k,k-1}(\theta, \varphi) dm_\lambda(\varphi) + \frac{1}{2\pi\Gamma(k)} \frac{1}{(\sin \theta)^\lambda} \\ &\times \frac{d^k}{d\theta^k} \int_0^\pi \frac{f(\varphi)}{(\sin \varphi)^\lambda} \int_0^1 \left(\log \frac{1}{r} \right)^{k-1} \left(\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} - 1 \right) \frac{dr}{r} dm_\lambda(\varphi) \end{aligned}$$

$$\begin{aligned}
&= -\frac{\lambda \cos \theta}{(\sin \theta)^{\lambda+1}} \int_0^\pi \frac{f(\varphi)}{(\sin \varphi)^\lambda} R^{k,k-1}(\theta, \varphi) dm_\lambda(\varphi) \\
&\quad + \frac{1}{(\sin \theta)^\lambda} \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta-\varphi| > \varepsilon}^\pi \frac{f(\varphi)}{(\sin \varphi)^\lambda} R^k(\theta, \varphi) dm_\lambda(\varphi) + \gamma_k f(\theta),
\end{aligned}$$

where the integral after the last equal sign is absolutely convergent and $\gamma_k = 0$, if k is odd and $\gamma_k = (-1)^{\frac{k}{2}}$, when k is even.

A careful study of Lemma 2.1 and again a distributional argument allow us to justify the differentiation under the integral sign (see Lemma 4.2 in Appendix) to get

$$\begin{aligned}
(26) \quad &\frac{d}{d\theta} \int_0^\pi f(\varphi) \left(R_\lambda^{k,k-1}(\theta, \varphi) - \frac{R^{k,k-1}(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} \right) dm_\lambda(\varphi) = \frac{\lambda \cos \theta}{(\sin \theta)^{\lambda+1}} \int_0^\pi \frac{f(\varphi)}{(\sin \varphi)^\lambda} R^{k,k-1}(\theta, \varphi) dm_\lambda(\varphi) \\
&\quad + \int_0^\pi f(\varphi) \left(R_\lambda^k(\theta, \varphi) - \frac{R^k(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} \right) dm_\lambda(\varphi), \quad \text{a.e. } \theta \in (0, \pi),
\end{aligned}$$

where all the integrals are absolutely convergent.

By combining (25) and (26) we conclude that

$$\begin{aligned}
\frac{d^k}{d\theta^k} L_\lambda^{-\frac{k}{2}} f(\theta) &= \frac{d}{d\theta} \int_0^\pi f(\varphi) \left(R_\lambda^{k,k-1}(\theta, \varphi) - \frac{R^{k,k-1}(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} \right) dm_\lambda(\varphi) \\
&\quad + \frac{d}{d\theta} \int_0^\pi f(\varphi) \frac{R^{k,k-1}(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} dm_\lambda(\varphi) \\
&= \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta-\varphi| > \varepsilon}^\pi f(\varphi) R_\lambda^k(\theta, \varphi) dm_\lambda(\varphi) - \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta-\varphi| > \varepsilon}^\pi f(\varphi) \frac{R^k(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} dm_\lambda(\varphi) \\
&\quad + \lambda \frac{\cos \theta}{(\sin \theta)^{\lambda+1}} \int_0^\pi \frac{f(\varphi)}{(\sin \varphi)^\lambda} R^{k,k-1}(\theta, \varphi) dm_\lambda(\varphi) + \frac{d}{d\theta} \int_0^\pi f(\varphi) \frac{R^{k,k-1}(\theta, \varphi)}{(\sin \theta \sin \varphi)^\lambda} dm_\lambda(\varphi) \\
&= \lim_{\varepsilon \rightarrow 0^+} \int_{0, |\theta-\varphi| > \varepsilon}^\pi f(\varphi) R_\lambda^k(\theta, \varphi) dm_\lambda(\varphi) + \gamma_k f(\theta), \quad \text{a.e. } \theta \in (0, \pi).
\end{aligned}$$

Thus the proof of Theorem 1.1 is complete.

3. PROOF OF THEOREM 1.2

In order to show Theorem 1.2 we need to improve Lemma 2.1 as follows,

Lemma 3.1. *Let $k \in \mathbb{N}$. If R^k and A_2 are defined as in Lemma 2.1, then*

$$R^k(\theta, \varphi) = M_k \frac{1}{\sin(\theta - \varphi)} + O\left(\sqrt{\frac{1}{|\theta - \varphi|}}\right), \quad (\theta, \varphi) \in A_2, \quad \theta \neq \varphi,$$

for a certain $M_k \in \mathbb{R}$. Moreover, $M_k = 0$ provided that k is even.

Proof. According to (6) with $\lambda = t = 0$ we have that

$$\frac{\partial}{\partial \theta^k} \left(\frac{1}{\Delta_r} \right) = \sum_{s,i,j} c_{k,s,i,j} \frac{r^{i+j} a^i b^j}{\Delta_r^{1+s}},$$

where $a = \cos(\theta - \varphi)$, $b = -\sin(\theta - \varphi)$, and $c_{k,s,i,j} \neq 0$ only if $s = 1, \dots, k$, $j \geq 2s - k$ and $i + j = s$.

Note firstly that

$$\int_0^{\frac{1}{2}} \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \left| \frac{\partial}{\partial \theta^k} \left(\frac{1}{1+r^2-2r\cos(\theta-\varphi)} \right) \right| \frac{dr}{r} \leq C, \quad \theta, \varphi \in (0, \pi).$$

Also, we have

$$\begin{aligned} \int_{\frac{1}{2}}^1 \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{r^{i+j} |a|^i |b|^j}{\Delta_r^{1+s}} \frac{dr}{r} &\leq C \int_{\frac{1}{2}}^1 \frac{(1-r)^k |b|^j}{((1-r)^2 + \Delta)^{1+s}} dr \\ &\leq C \frac{|b|^j}{\Delta^{s-\frac{k}{2}+\frac{1}{2}}} \int_0^{\frac{1}{2\sqrt{\Delta}}} \frac{u^k}{(1+u)^{2+2s}} du \\ &\leq C \frac{|b|^j}{\Delta^{\frac{j}{2}+\frac{1}{4}}} \int_0^{\frac{1}{2\sqrt{\Delta}}} \frac{u^{2s-j+\frac{1}{2}}}{(1+u)^{2+2s}} du \\ &\leq C \frac{1}{\sqrt{|\theta-\varphi|}}, \quad (\theta, \varphi) \in A_2, \quad \theta \neq \varphi, \end{aligned}$$

provided that $s = 1, \dots, k$, $i + j = s$, $j > 2s - k$.

Assume now $s = 1, \dots, k$, $i + j = s$, $j = 2s - k$. By using the mean value theorem we get

$$\begin{aligned} \int_{\frac{1}{2}}^1 \left(\log \frac{1}{r} \right)^{k-1} (1-r^2) \frac{r^{i+j} a^i b^j}{\Delta_r^{1+s}} \frac{dr}{r} \\ = 2 \int_{\frac{1}{2}}^1 \frac{(1-r)^k}{((1-r)^2 + \Delta)^{1+s}} dr a^i b^j + O \left(\frac{1}{\sqrt{|\theta-\varphi|}} \right), \quad \theta, \varphi \in A_2, \quad \theta \neq \varphi. \end{aligned}$$

Moreover, since $2s - k \geq 0$,

$$\int_{\frac{1}{2}}^1 \frac{(1-r)^k}{((1-r)^2 + \Delta)^{1+s}} dr = \frac{1}{\Delta^{s-\frac{k}{2}+\frac{1}{2}}} \left(\int_0^\infty \frac{u^k du}{(1+u^2)^{1+s}} - \int_{\frac{1}{2\sqrt{\Delta}}}^\infty \frac{u^k du}{(1+u^2)^{1+s}} \right)$$

and

$$\begin{aligned} \frac{|a|^i |b|^j}{\Delta^{s-\frac{k}{2}+\frac{1}{2}}} \int_{\frac{1}{2\sqrt{\Delta}}}^\infty \frac{u^k du}{(1+u^2)^{1+s}} &\leq C \frac{1}{\Delta^{1/4}} \int_{\frac{1}{2\sqrt{\Delta}}}^\infty \frac{u^{k+\frac{1}{2}} du}{(1+u)^{2+s}} \\ &\leq \frac{C}{\Delta^{\frac{1}{4}}} \leq \frac{C}{|\theta-\varphi|^{\frac{1}{2}}}, \quad \theta, \varphi \in A_2, \quad \theta \neq \varphi. \end{aligned}$$

Also, if k is odd, we have that

$$\frac{a^i b^j}{\Delta^{s-\frac{k}{2}+\frac{1}{2}}} = \frac{(\cos(\theta - \varphi))^i (-\sin(\theta - \varphi))^j}{(2(1 - \cos(\theta - \varphi)))^{\frac{i}{2}+\frac{1}{2}}} = -\frac{1}{\sin(\theta - \varphi)} + O\left(\frac{1}{|\theta - \varphi|^{\frac{1}{2}}}\right), \quad \theta, \varphi \in A_2, \theta \neq \varphi.$$

By combining the above estimates we conclude that

$$R^k(\theta, \varphi) = M_k \frac{1}{\sin(\theta - \varphi)} + O\left(\frac{1}{|\theta - \varphi|^{\frac{1}{2}}}\right), \quad \theta, \varphi \in A_2, \theta \neq \varphi,$$

for a certain $M_k \in \mathbb{R}$, for every k odd.

Assume now that k is even. We get

$$\frac{a^i b^j}{\Delta^{s-\frac{k}{2}+\frac{1}{2}}} = \frac{1}{|\sin(\theta - \varphi)|} + O\left(\frac{1}{|\theta - \varphi|^{\frac{1}{2}}}\right), \quad \theta, \varphi \in A_2, \theta \neq \varphi.$$

Hence, from Lemma 2.1 we deduce that, for every $\theta, \varphi \in A_2, \theta \neq \varphi$,

$$R_\lambda^k(\theta, \varphi) = M_k \frac{1}{|\sin(\theta - \varphi)|(\sin \theta \sin \varphi)^\lambda} + O\left(\frac{1}{(\sin \theta \sin \varphi)^{\lambda+1/2}} \left(1 + \sqrt{\frac{\sin \theta}{|\theta - \varphi|}}\right)\right).$$

By virtue of Theorem 1.1, $M_k = 0$ because

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\theta/2, |\theta - \varphi| > \varepsilon}^{3\theta/2} \frac{1}{|\sin(\theta - \varphi)|} (\sin \varphi)^\lambda d\varphi,$$

does not exist for every $\theta \in (0, \pi/2)$. □

From Lemmas 2.1 and 3.1 we deduce that,

$$(27) \quad R_\lambda^k(\theta, \varphi) = \begin{cases} O((\sin \varphi)^{-(2\lambda+1)}), & (\theta, \varphi) \in A_1; \\ \frac{M_k}{(\sin \theta \sin \varphi)^\lambda \sin(\theta - \varphi)} + O\left(\frac{1}{(\sin \varphi)^{2\lambda+1}} \left(1 + \sqrt{\frac{\sin \varphi}{|\theta - \varphi|}}\right)\right), & (\theta, \varphi) \in A_2, \theta \neq \varphi; \\ O((\sin \theta)^{-(2\lambda+1)}), & (\theta, \varphi) \in A_3. \end{cases}$$

By using (27) we can prove Theorem 1.2 by proceeding as in the proof of [1, Proposition 8.1].

4. APPENDIX

In this appendix we present the results we need about differentiation under the integral sign. We think that these results are wellknown but we have not found a exact reference (only the unpublished notes [5]). Then we prefer to include here a proof of the result in the form we use. We look for conditions on a function f defined on $\mathbb{R} \times \mathbb{R}$ in order that the formula

$$\frac{\partial}{\partial x} \int_{\mathbb{R}} f(x, y) dy = \int_{\mathbb{R}} \frac{\partial}{\partial x} f(x, y) dy, \quad \text{a.e. } x \in \mathbb{R},$$

holds.

In the following we establish conditions on a function f in order that distributional and classical derivatives of f coincide.

Lemma 4.1. *Let $-\infty \leq a < b \leq +\infty$. Assume that f is a continuous function on $I \times I$, where $I = (a, b)$, such that*

(i) *For every $y \in I$, the function $\frac{\partial}{\partial x}f(x, y)dy$ is continuous on $I \setminus \{y\}$, where the derivative is understood in the classical sense.*

(ii) *For every $y \in I$ and every compact subset K of I , $\int_K |f(x, y)|dx < +\infty$, and*

$$\int_K \left| \frac{\partial f}{\partial x}(x, y) \right| dx < +\infty.$$

Then, $D_x f(x, y) = \frac{\partial}{\partial x}f(x, y)$, for every $y \in I$. Here, as above, $D_x f(x, y)$ denotes the distributional derivative respect to x of f .

Proof. Let $g \in C_c^\infty(I)$. We can write

$$\begin{aligned} \langle D_x f(x, y), g(x) \rangle &= - \lim_{\varepsilon \rightarrow 0^+} \left(\int_a^{y-\varepsilon} + \int_{y+\varepsilon}^b \right) g'(x) f(x, y) dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \left[-g(y-\varepsilon)f(y-\varepsilon, y) + g(y+\varepsilon)f(y+\varepsilon, y) + \left(\int_a^{y-\varepsilon} + \int_{y+\varepsilon}^b \right) g(x) \frac{\partial f}{\partial x}(x, y) dx \right] \\ &= \int_a^b g(x) \frac{\partial f}{\partial x}(x, y) dx, \quad y \in I. \end{aligned}$$

Then, $D_x f(x, y) = \frac{\partial f}{\partial x}(x, y)$, $y \in I$. □

The differentiations under the integral sign that we have made in the proof of our results can be justified by using the following one.

Lemma 4.2. *Suppose that f is a measurable function defined on $\mathbb{R} \times \mathbb{R}$ that satisfies the following conditions:*

(i) *for every compact subset K of \mathbb{R} , $\int_K \int_{\mathbb{R}} |f(x, y)| dy dx < \infty$, and*

(ii) *there exists a measurable function g on $\mathbb{R} \times \mathbb{R}$ such that $\int_K \int_{\mathbb{R}} |g(x, y)| dy dx < \infty$, for every compact subset K of \mathbb{R} , and that the distributional derivative $D_x f(\cdot, y)$ is represented by $g(\cdot, y)$, for every $y \in \mathbb{R}$.*

Then,

$$\frac{\partial}{\partial x} \int_{\mathbb{R}} f(x, y) dy = \int_{\mathbb{R}} \frac{\partial}{\partial x} f(x, y) dy, \quad a.e. \ x \in \mathbb{R},$$

where the derivatives are understood in the classical sense.

Proof. We define the function $h(x) = \int_{\mathbb{R}} f(x, y) dy$, $x \in \mathbb{R}$. By (i) h defines a regular distribution that we continue denoting by h . According to [12, Chap. 2, §5, Theorem V], we have that

$$\frac{\partial}{\partial x} f(x, y) = g(x, y), \quad \text{a.e. } (x, y) \in \mathbb{R} \times \mathbb{R},$$

where the derivative is understood in the classical sense. Moreover, if $F \in C_c^\infty(\mathbb{R})$, then

$$\langle D_x h, F \rangle = \int_{\mathbb{R}} F(x) \int_{\mathbb{R}} \frac{\partial f}{\partial x}(x, y) dy dx.$$

Hence, $D_x h(x) = \int_{\mathbb{R}} \frac{\partial f}{\partial x}(x, y) dy$ in the distributional sense. By using again [12, Chap. 2, §5, Theorem V] we conclude that

$$\frac{\partial}{\partial x} h(x) = \int_{\mathbb{R}} \frac{\partial}{\partial x} f(x, y) dy, \quad \text{a.e. } x \in \mathbb{R}.$$

Thus the proof is completed. □

REFERENCES

- [1] D. Buraczewski, T. Martínez, J. L. Torrea, and R. Urban, *On the Riesz transform associated with the ultraspherical polynomials*, J. Anal. Math. **98** (2006), 113–143.
- [2] D. Buraczewski, T. Martínez, and J. L. Torrea, *Calderón-Zygmund operators associated to ultraspherical expansions*, Canad. J. Math. **59** (2007), no. 6, 1223–1244.
- [3] J. T. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, *Oscillation and variation for the Hilbert transform*, Duke Math. J. **105** (2000), no. 1, 59–83.
- [4] J. T. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, *Oscillation and variation for singular integrals in higher dimensions*, Trans. Amer. Math. Soc. **355** (2003), no. 5, 2115–2137.
- [5] S. Cheng, *Differentiation under the integral sign using weak derivatives*, unpublished manuscript. <http://www.gold-saucer.org/math>
- [6] T. A. Gillespie and J. L. Torrea, *Dimension free estimates for the oscillation of Riesz transforms*, Israel J. Math. **141** (2004), 125–144.
- [7] L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Inc., Upper Saddle River, N.J., 2004.
- [8] R. L. Jones and G. Wang, *Variation inequalities for the Fejér and Poisson kernels*, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4493–4518.
- [9] B. Muckenhoupt, *Transplantation theorems and multiplier theorems for Jacobi series*, Mem. Amer. Math. Soc. **64** (1986), no. 356.
- [10] B. Muckenhoupt and E. M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc. **118** (1965), 17–92.

- [11] S. Roman, *The formula of Faà di Bruno*, Amer. Math. Monthly **87** (1980), 805–809.
- [12] L. Schwartz, *Théorie des distributions*, Hermann, París, 1973.
- [13] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory.*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J., 1970.
- [14] G. Szegő, *Orthogonal polynomials*, fourth ed., American Mathematical Society, Providence, R.I., 1975, American Mathematical Society, Colloquium Publications, Vol. XXIII.

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, CAMPUS DE ANCHIETA, AVDA. ASTROFÍSICO FRANCISCO SÁNCHEZ, s/N, 38271 LA LAGUNA (STA. CRUZ DE TENERIFE), SPAIN

E-mail address: jbetanco@ull.es; jcfarina@ull.es; lrguez@ull.es

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD NACIONAL DEL SUR, AVDA. ALEM 1253 - 2 PISO, 8000 BAHÍA BLANCA, BUENOS AIRES, ARGENTINA

E-mail address: ricardo.testoni@uns.edu.ar